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ON SOME FORMULAS FOR THE GENERALIZED APPELL TYPE FUNCTIONS

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Abstract. A remarkably large number of special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been investigated by many authors. Motivated the works of both works of both Burchnall and Chaundy and Chaundy and very recently, Brychkov and Saad gave interesting generalizations of Appell type functions. In the present sequel to the aforementioned investigations and some of the earlier works listed in the reference, we present some new differential formulas for the generalized Appell's type functions $\kappa_i, i = 1, 2, \ldots, 18$ by considering the product of two ${}_4F_3$ functions.

1. Introduction

During last four decades or so, the theories of familiar special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on)get great success through the research contributions of various authors (see [1–6, 14, 16, 21, 22]; for recent developments, see also [7, 8, 12]).

Motivated the works of both Burchnall and Chaundy [9, 10] and Chaundy [11], very recently, Choi *et al.* [13] gave certain interesting generalization of the Appell type functions and study their properties. In the present sequel to the aforementioned investigations, we present some new differential formulas for the generalized Appell's type functions κ_i, $i = 1, 2, \ldots, 18$ by considering the product of two ${}_4F_3$ functions.

For our purpose, we begin by recalling Burchnall and Chaundy [9, 10] suggested a possible extension of their results to functions of higher order (with more parameters) for two variable as follows:

(1.1)

$$ {}_{p+1}F_p^{(2)}\left[\begin{array}{l} a;\ b_1, b_2, \ldots,\ b_p; b_1^{'}, b_2^{'}, \ldots,\ b_p^{'}\ ; \\ \quad c_1, c_2, \ldots,\ c_p; c_1^{'}, c_2^{'}, \ldots,\ c_p^{'}\ ; \end{array}\quad x,\ y\right] $$

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$$= \sum_{m,\,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m \cdots (b_p)_m\ (b_1^{'})_n(b_2^{'})_n \cdots\ (b_p^{'})_n}{m!n!(c_1)_m(c_2)_m \cdots\ (c_p)_m(c_1^{'})_n(c_2^{'})_n \cdots\ (c_p^{'})_n}\ x^m\ y^n$$

$$= \nabla(a)\ {}_{p+1}F_p\left[\begin{matrix} a,\ b_1, b_2, \ldots,\ b_p; \\ c_1, c_2, \ldots,\ c_p; \end{matrix}\quad x\right]\ {}_{p+1}F_p\left[\begin{matrix} a,\ b_1^{'}, b_2^{'}, \ldots,\ b_p^{'}; \\ c_1^{'}, c_2^{'}, \ldots,\ c_p^{'}; \end{matrix}\quad y\right]$$

$$(1.2)\qquad {}_{p+1}F_p^{(2)}\left[\begin{matrix} a;\ b_1, b_2,\ \ldots,\ b_p; b_1^{'}, b_2^{'}, \ldots,\ b_p^{'}\ ; \\ c_1, c_2,\ \ldots,\ c_p; c_1^{'}, c_2^{'},\ \ldots,\ c_p^{'}\ ; \end{matrix}\quad x,\ y\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b_1)_r \cdots (b_p)_r\ (b_1^{'})_r(b_2^{'})_r \cdots\ (b_p^{'})_r}{r!(c_1)_r(c_2)_r \cdots\ (c_p)_r(c_1^{'})_r(c_2^{'})_r \cdots\ (c_p^{'})_r}\ x^r\ y^r$$

$$\times\ {}_{p+1}F_p\left[\begin{matrix} a+r,\ b_1+r, b_2+r, \ldots,\ b_p+r; \\ c_1+r, c_2+r, \ldots,\ c_p+r; \end{matrix}\quad x\right]$$

$$\times\ {}_{p+1}F_p\left[\begin{matrix} a+r,\ b_1^{'}+r, b_2^{'}+r, \ldots,\ b_p^{'}+r; \\ c_1^{'}, c_2^{'}, \ldots,\ c_p^{'}; \end{matrix}\quad y\right].$$

Recently, Khan and Abukhammash [15] introduced 10 Appell type generalized functions M_i $(i = 1, \ldots, 10)$ by considering the product of two ${}_3F_2$ functions as follows.

$$(1.3)\qquad M_1\left(a, a^{'}, b, b^{'}, c, c^{'}; d, e, e^{'}; x, y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m \left(a^{'}\right)_n (b)_m \left(b^{'}\right)_n (c)_m \left(c^{'}\right)_n}{(d)_{m+n}\,(e)_m\,(e^{'})_n} \frac{x^m}{n!} \frac{y^n}{n!},$$

$$(1.4)\qquad M_2\left(a, a^{'}, b, b^{'}, c, c^{'}; d, e; x, y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m \left(a^{'}\right)_n (b)_m \left(b^{'}\right)_n (c)_m \left(c^{'}\right)_n}{(d)_{m+n}\,(e)_{m+n}} \frac{x^m}{n!} \frac{y^n}{n!},$$

$$(1.5)\qquad M_3\left(a, b, b^{'}, c, c^{'}; d, d^{'} e, e^{'}; x, y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m \left(b^{'}\right)_n (c)_m \left(c^{'}\right)_n}{(d)_m\,(d^{'})_n\,, (e)_m\,(e^{'})_n} \frac{x^m}{n!} \frac{y^n}{n!},$$

$$(1.6)\qquad M_4\left(a, b, b^{'}, c, c^{'}; d, e, e^{'}; x, y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m \left(b^{'}\right)_n (c)_m \left(c^{'}\right)_n}{(d)_{m+n}\,(e)_m\,(e^{'})_n} \frac{x^m}{n!} \frac{y^n}{n!},$$

$$(1.7)\qquad M_5\left(a, b, b^{'}, c, c^{'}; d, e; x, y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\left(b^{'}\right)_n(c)_m\left(c^{'}\right)_n}{(d)_{m+n}\,(e)_{m+n}}\frac{x^m}{n!}\frac{y^n}{n!},$$

(1.8)
$$M_6\left(a,b,c,c^{'};d,d^{'},e,e^{'};x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\,(c)_m\left(c^{'}\right)_n}{(d)_m\,(d^{'})_n\,(e)_m\,(e^{'})_n}\frac{x^m}{n!}\frac{y^n}{n!},$$

(1.9)
$$M_7\left(a,b,c,c^{'};d,e,e^{'};x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\,(c)_m\left(c^{'}\right)_n}{(d)_{m+n}\,(e)_m\,(e)_n}\frac{x^m}{n!}\frac{y^n}{n!},$$

(1.10)
$$M_8\left(a,b,c,c^{'};d,e;x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\left(b^{'}\right)_n\left(c^{'}\right)_n}{(d)_{m+n}\,(e)_{m+n}}\frac{x^m}{n!}\frac{y^n}{n!},$$

(1.11)
$$M_9\left(a,b,c;d,d^{'},e,e^{'};x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\,(c)_m\left(c^{'}\right)_n}{(d)_m\,(d^{'})_n\,(e)_m\,(e^{'})_n}\frac{x^m}{n!}\frac{y^n}{n!},$$

(1.12)
$$M_{10}\left(a,b^{'},c;d,e,e^{'};x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\,(c)_{m+n}}{(d)_{m+n}\,(e)_m\,(e^{'})_n}\frac{x^m}{n!}\frac{y^n}{n!}.$$

In this paper, we consider the product of two ${}_4F_3$ hypergeometric functions, i.e.,

(1.13)
$${}_4F_3(a,b,c,d;e,f,g;x)\,{}_4F_3(a',b',c',d';e',f',g';y) = \sum_{m,\,n=0}^{\infty} \frac{(a)_m\;(a')_n\;(b)_m\;(b')_n\;(c)_m\;(c')_n\;(d)_m\;(d')_n}{(e)_m\;(e')_n\;(f)_m\;(f')_n\;(g)_m\;(g')_n}\frac{x^m}{m!}\;\frac{y^n}{n!}.$$

This double series, in itself, yields nothing new, but by replacing one or more of the seven pairs of products

$$(a)_m(a')_n, \quad (b)_m(b')_n, \quad (c)_m(c')_n,$$

$$(d)_m(d')_n, \quad (e)_m(e')_n, \quad (f)_m(f')_n, \quad (g)_m(g')_n$$

by the corresponding expressions

$$(a)_{m+n},\ (b)_{m+n},\ (c)_{m+n},\ (d)_{m+n},\ (e)_{m+n},\ (f)_{m+n},$$

we are led to nineteen distinct possibilities of getting new double series. One such possibility, however, gives us the double series

$$\sum_{m,\,n=0}^{\infty} \frac{(a)_{m+n}\ (b)_{m+n}\ (c)_{m+n}\ (d)_{m+n}}{(e)_{m+n}\ (f)_{m+n}\ (g)_{m+n}}\ \frac{x^m}{m!}\frac{y^n}{n!},$$

which, upon using the well-known (easily-derivable) identity (see, *e.g.*, [19]):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\,\frac{(x+y)^N}{N!}, \tag{1.14}$$

is simply the hypergeometric series ${}_4F_3(a,b,c,d;e,f,g;x+y)$.

The remaining possibilities lead to the following eighteen generalized Appell type functions of two variables (see also [13]):

$$\begin{aligned} &\kappa_1(a,a',b,b',c,c',d,d'\ ;e,f,f',g,g';x,y) \\ &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!} \\ &(\max\{|x|,\ |y|\}<1)\,; \end{aligned} \tag{1.15}$$

$$\begin{aligned} &\kappa_2(a,a',b,b',c,c',d,d'\ ;e,f,g,g';x,y) \\ &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!} \\ &(\max\{|x|,\ |y|\}<1)\,; \end{aligned} \tag{1.16}$$

$$\begin{aligned} &\kappa_3(a,a',b,b',c,c',d,d'\ ;e,f,g;x,y) \\ &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_{m+n}}\ \frac{x^m}{m!}\ \frac{y^n}{n!} \\ &(\max\{|x|,\ |y|\}<1)\,; \end{aligned} \tag{1.17}$$

$$\begin{aligned} &\kappa_4(a,b,b',c,c',d,d'\ ;e,e',f,f',g,g';x,y) \\ &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_m(e')_n(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!} \\ &(|x|+|y|<1)\,; \end{aligned} \tag{1.18}$$

$$\begin{aligned} &\kappa_5(a,b,b',c,c',d,d'\ ;e,f,f',g,g';x,y) \\ &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!} \\ &(\max\{|x|,\ |y|\}<1)\,; \end{aligned} \tag{1.19}$$

$$\begin{aligned}\kappa_6(a,b,b',c,c',d,d'\ ;e,f,g,g';x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ (\max\{|x|,\ |y|\}<1)\,;\end{aligned} \tag{1.20}$$

$$\begin{aligned}\kappa_7(a,b,b',c,c',d,d'\ ;e,f,g;x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_{m+n}}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ (\max\{|x|,\ |y|\}<1)\,;\end{aligned} \tag{1.21}$$

$$\begin{aligned}\kappa_8(a,b,,c,c',d,d'\ ;e,e',f,f',g,g';x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n(d)_m(d')_n}{(e)_m(e')_n(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ \left(\sqrt{|x|}+\sqrt{|y|}<1\right);\end{aligned} \tag{1.22}$$

$$\begin{aligned}\kappa_9(a,b,c,c',d,d'\ ;e,f,f',g,g';x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ (|x|+|y|<1)\,;\end{aligned} \tag{1.23}$$

$$\begin{aligned}\kappa_{10}(a,b,c,c',d,d'\ ;e,f,g,g';x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ (\max\{|x|,\ |y|\}<1)\,;\end{aligned} \tag{1.24}$$

$$\begin{aligned}\kappa_{11}(a,b,c,c',d,d'\ ;e,f,g;x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_{m+n}}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ (\max\{|x|,\ |y|\}<1)\,;\end{aligned} \tag{1.25}$$

$$\begin{aligned}\kappa_{12}(a,b,c,d,d'\ ;e,e',f,f',g,g';x,y)\\ := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_m(d')_n}{(e)_m(e')_n(f)_m(f')_n(g)_m(g')_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}\\ \left(\sqrt[3]{|x|}+\sqrt[3]{|y|}<1\right);\end{aligned} \tag{1.26}$$

$$\kappa_{13}(a,b,c,d,d'\ ;e,f,f',g,g';x,y) \tag{1.27}$$

$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_m(d')_n}{(e)_{m+n}(f)_m(f')_n(g)_m(g')_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$\left(\sqrt{|x|}+\sqrt{|y|}<1\right);$$

(1.28)
$$\kappa_{14}(a,b,c,d,d'\;;e,f,g,g';x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_m(g')_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$(|x|+|y|<1);$$

(1.29)
$$\kappa_{15}(a,b,c,d,d'\;;e,f,g;x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_m(d')_n}{(e)_{m+n}(f)_{m+n}(g)_{m+n}} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$(\max\{|x|,\,|y|\}<1);$$

(1.30)
$$\kappa_{16}(a,b,c,d;e,e',f,f',g,g';x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_{m+n}}{(e)_m(e')_n(f)_m(f')_n(g)_m(g')_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$\left(\sqrt[4]{|x|}+\sqrt[4]{|y|}<1\right);$$

(1.31)
$$\kappa_{17}(a,b,,c,d;e,f,f',g,g';x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_{m+n}}{(e)_{m+n}(f)_m(f')_n(g)_m(g')_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$\left(\sqrt[3]{|x|}+\sqrt[3]{|y|}<1\right);$$

(1.32)
$$\kappa_{18}(a,b,,c,d;e,f,g,g';x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_{m+n}}{(e)_{m+n}(f)_{m+n}(g)_m(g')_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!}$$
$$\left(\sqrt{|x|}+\sqrt{|y|}<1\right).$$

2. Fractional derivatives

In 1971 Euler extended the derivative formula

$$D_z^n\{z^\lambda\} = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)z^{\lambda-n}$$

(2.1)
$$= \frac{\Gamma\,(1+\lambda)}{\Gamma\,(1+\lambda-n)} z^{\lambda-n} \; (n=0,1,2,3,\cdots)$$

to the general form

$$D_z^{\mu}\{z^{\lambda}\} = \frac{\Gamma\,(1+\lambda)}{\Gamma\,(1+\lambda-\mu)} z^{\lambda-\mu} \tag{2.2}$$

where μ is an arbitrary complex number.

Theorem 1. *Each of the following integral formulas holds true:*

$$\begin{aligned} D_z^{\lambda-\mu}&\left\{z^{\lambda-1}\ {}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma;\\ \delta,\eta;\end{matrix}\quad az\right]\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\ {}_4F_3\left[\begin{matrix}\alpha,\beta,\gamma,\lambda;\\ \delta,\eta,\mu;\end{matrix}\quad az\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} D_z^{\lambda-\mu}&\left\{z^{\lambda-1}\ {}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma;\\ \delta,\eta;\end{matrix}\quad xz\right]\ {}_3F_2\left[\begin{matrix}\alpha',\beta',\gamma';\\ \delta',\eta'\ ;\end{matrix}\quad yz\right]\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\kappa_6(\lambda,\alpha,\beta,\gamma,\alpha',\beta',\gamma';\mu,\delta,\delta',\eta,\eta';xz,yz), \end{aligned} \tag{2.4}$$

$$\begin{aligned} D_z^{a-e}&\left\{z^{a-1}M_1\left[\begin{matrix}b,b',c,c',d,d';\\ f,g,g';\end{matrix}\quad xz,yz\right]\right\} \\ &= \frac{\Gamma(a)}{\Gamma(e)} z^{e-1}\ \kappa_6\left[\begin{matrix}a,b,b',c,c',d,d';\\ e,f,g,g';\end{matrix}\quad xz,yz\right], \end{aligned} \tag{2.5}$$

$$\begin{aligned} D_z^{a-e}&\left\{z^{a-1}M_2\left[\begin{matrix}b,b',c,c',d,d';\\ f,g;\end{matrix}\quad xz,yz\right]\right\} \\ &= \frac{\Gamma(a)}{\Gamma(e)} z^{e-1}\ \kappa_7\left[\begin{matrix}a,b,b',c,c',d,d';\\ e,f,g;\end{matrix}\quad xz,yz\right], \end{aligned} \tag{2.6}$$

$$\begin{aligned} D_z^{a-e}&\left\{z^{a-1}M_3\left[\begin{matrix}b,c,c',d,d';\\ f,f',g,g';\end{matrix}\quad xz,yz\right]\right\} \\ &= \frac{\Gamma(a)}{\Gamma(e)} z^{e-1}\ \kappa_9\left[\begin{matrix}a,b,c,c',d,d';\\ e,f,f',g,g';\end{matrix}\quad xz,yz\right], \end{aligned} \tag{2.7}$$

$$\begin{aligned} D_z^{a-e}&\left\{z^{a-1}M_4\left[\begin{matrix}b,c,c',d,d';\\ f,g,g';\end{matrix}\quad xz,yz\right]\right\} \\ &= \frac{\Gamma(a)}{\Gamma(e)} z^{e-1}\ \kappa_{10}\left[\begin{matrix}a,b,c,c',d,d';\\ e,f,g,g';\end{matrix}\quad xz,yz\right], \end{aligned} \tag{2.8}$$

$$\begin{aligned} D_z^{a-e}&\left\{z^{a-1}M_5\left[\begin{matrix}b,c,c',d,d';\\ f,g;\end{matrix}\quad xz,yz\right]\right\} \\ &= \frac{\Gamma(a)}{\Gamma(e)} z^{e-1}\ \kappa_{11}\left[\begin{matrix}a,b,c,c',d,d';\\ e,f,g;\end{matrix}\quad xz,yz\right], \end{aligned} \tag{2.9}$$

$$D_z^{a-e}\left\{z^{a-1}M_6\left[\begin{array}{c}b,c,d,d';\\ f,f',g,g';\end{array}\quad xz,yz\right]\right\}$$
$$=\frac{\Gamma(a)}{\Gamma(e)}z^{e-1}\ \kappa_{13}\left[\begin{array}{c}a,b,c,d,d';\\ e,f,f',g,g';\end{array}\quad xz,yz\right], \tag{2.10}$$

$$D_z^{a-e}\left\{z^{a-1}M_7\left[\begin{array}{c}b,c,d,d';\\ f,g,g';\end{array}\quad xz,yz\right]\right\}$$
$$=\frac{\Gamma(a)}{\Gamma(e)}z^{e-1}\ \kappa_{14}\left[\begin{array}{c}a,b,c,d,d';\\ e,f,g,g';\end{array}\quad xz,yz\right], \tag{2.11}$$

$$D_z^{a-e}\left\{z^{a-1}M_8\left[\begin{array}{c}b,c,d,d';\\ f,g;\end{array}\quad xz,yz\right]\right\}$$
$$=\frac{\Gamma(a)}{\Gamma(e)}z^{e-1}\ \kappa_{15}\left[\begin{array}{c}a,b,c,d,d';\\ e,f,g;\end{array}\quad xz,yz\right], \tag{2.12}$$

$$D_z^{a-e}\left\{z^{a-1}M_9\left[\begin{array}{c}b,c,d;\\ f,f',g,g';\end{array}\quad xz,yz\right]\right\}$$
$$=\frac{\Gamma(a)}{\Gamma(e)}z^{e-1}\ \kappa_{17}\left[\begin{array}{c}a,b,c,d;\\ e,f,f',g,g';\end{array}\quad xz,yz\right], \tag{2.13}$$

$$D_z^{a-e}\left\{z^{a-1}M_{10}\left[\begin{array}{c}b,c,d;\\ f,g,g';\end{array}\quad xz,yz\right]\right\}$$
$$=\frac{\Gamma(a)}{\Gamma(e)}z^{e-1}\ \kappa_{18}\left[\begin{array}{c}a,b,c,d;\\ e,f,g,g';\end{array}\quad xz,yz\right], \tag{2.14}$$

where M_i's are given in [15].

Proof. Using the newly introduced functions κ_i and the functions M_i in [15] with the aid of the formulas (2.1) and (2.2), we can easily derive the results (2.3)-(2.14). So details of proofs are omitted. □

2.1. A set of lemmas

Burchnall and Chaundy [9, 10] quote as lemmas the well known identities

$$\frac{\Gamma(h)\Gamma(m+n+h)}{\Gamma\ (m+h)\Gamma(n+h)}=\sum_{r=0}^{\infty}\frac{(-m)_r(-n)_r}{r!(h)_r}, \tag{2.15}$$

$$\frac{\Gamma(m+h)\Gamma(n+h)}{\Gamma(h)\Gamma(m+n+h)}=\sum_{r=0}^{\infty}\frac{(-m)_r(-n)_r}{r!(-h-m-n+1)_r}, \tag{2.16}$$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(h)_{2r}(-m)_r(-n)_r}{r!(-h+r-1)_r(m+h)_r(n+h)_r}, \tag{2.17}$$

$$\frac{\Gamma(h)\Gamma(m+n+h)\Gamma\ (m+k)\Gamma\ (n+k)}{\Gamma\ (m+h)\Gamma\ (n+h)\Gamma(k)\Gamma(m+n+k)}$$

$$= \sum_{r=0}^{\infty} \frac{(k-h)_r (k)_{2r} (-m)_r (-n)_r}{r!(k+r-1)_r (m+k)_r (n+k)_r (h)_r}, \tag{2.18}$$

$$= \sum_{r=0}^{\infty} \frac{(h-k)_r (-m)_r (-n)_r}{r!(h)_r (-k-m-n+1)_r}, \tag{2.19}$$

of these (2.15) is Gauss's theorem (or Vandermonde's theorem if m, n are positive integers), and (2.16) is a variant of (2.15) valid only when one of m , n is an integer; (2.17), (2.18) are limiting form of Dougall's theorem given by Bailey [23] and (2.19) is Saalschutz's theorem [24] valid only when one of m, n is a positive integer.

3. Expansions

Following the method adopted by Burchnall and Chaundy [9, 10] we obtain the following expansions of $\kappa_i, i = 1, 2, \ldots, 18$:

(3.1)
$$\kappa_1 \left[\begin{array}{l} a, a', b, b', c, c', d, d'; \\ e, f, f', g, g'; \end{array} x, y \right]$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_r (c)_r (d)_r (a')_r (b')_r (c')_r (d')_r}{r!(e+r-1)_r (e)_{2r} (f)_r (f')_r (g)_r (g')_r} x^r y^r$$
$$\times \ {}_4F_3 \left[\begin{array}{l} a+r, b+r, c+r, d+r; \\ e+2r, f+r, g+r \ ; \end{array} x \right] {}_4F_3 \left[\begin{array}{l} a'+r, b'+r, c'+r, d'+r; \\ e+2r, f'+r, g'+r \ ; \end{array} y \right],$$

$$ {}_4F_3 \left[\begin{array}{l} a, b, c, d; \\ e, f, g \ ; \end{array} x \right] {}_4F_3 \left[\begin{array}{l} a, b', c', d'; \\ e', f', g' \ ; \end{array} y \right] \tag{3.2}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_r (c)_r (d)_r (a')_r (b')_r (c')_r (d')_r}{r!(e)_r (e')_r (f)_r (f')_r (g)_r (g')_r} x^r y^r$$
$$\times \ \kappa_4 \left[\begin{array}{l} a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+r, e'+r, f+r, f'+r, g+r, g'+r; \end{array} x, y \right],$$

(3.3)
$$\kappa_4 \left[\begin{array}{l} a, b, b', c, c', d, d'; \\ e, e', f, f', g, g'; \end{array} x, y \right]$$
$$= \sum_{r=0}^{\infty} \frac{(b)_r (b')_r (c)_r (c')_r (d)_r (d')_r}{r!(e)_r (e')_r (f)_r (f')_r (g)_r (g')_r} x^r y^r$$
$$\times \ {}_4F_3 \left[\begin{array}{l} a+r, b+r, c+r, d+r; \\ e+2r, f+r, g+r \ ; \end{array} x \right] {}_4F_3 \left[\begin{array}{l} a+r, b'+r, c'+r, d'+r; \\ e'+r, f'+r, g'+r \ ; \end{array} y \right],$$

$$ {}_4F_3 \left[\begin{array}{l} a, b, c, d; \\ e, f, g \ ; \end{array} x \right] {}_4F_3 \left[\begin{array}{l} a', b', c', d'; \\ e, f', g' \ ; \end{array} y \right] \tag{3.4}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(a')_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_r(e)_r(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_1 \left[\begin{array}{l} a+r, a'+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+r, f+r, f'+r, g+r, g'+r; \end{array} x, y \right],$$

$$\kappa_5 \left[\begin{array}{l} a, b, b', c, c', d, d'; \\ e, f, f', g, g'; \end{array} x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_1 \left[\begin{array}{l} a+r, a'+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{array} x, y \right], \tag{3.5}$$

$$\kappa_1 \left[\begin{array}{l} a, a, b, b', c, c', d, d'; \\ e, f, f', g, g'; \end{array} x, y \right] = \sum_{r=0}^{\infty} \frac{(-1)^r(a)_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_5 \left[\begin{array}{l} a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{array} x, y \right], \tag{3.6}$$

$$\kappa_6 \left[\begin{array}{l} a, b, b', c, c', d, d'; \\ e, f, g, g'; \end{array} x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_r(g')_r} x^r y^r \times \kappa_2 \left[\begin{array}{l} a+r, a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, f'+r, g+r, g'+r; \end{array} x, y \right], \tag{3.7}$$

$$\kappa_2 \left[\begin{array}{l} a, a, b, b', c, c', d, d'; \\ e, f, g, g'; \end{array} x, y \right] = \sum_{r=0}^{\infty} \frac{(-1)^r(a)_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_r(g')_r} x^r y^r \times \kappa_6 \left[\begin{array}{l} a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, f'+r, g+r, g'+r; \end{array} x, y \right], \tag{3.8}$$

$$\kappa_2 \left[\begin{array}{l} a, a', b, b', c, c', d, d'; \\ e, f, g, g'; \end{array} x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_r(a')_r(b)_r(b')_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_r(g)_{2r}} x^r y^r \times \kappa_3 \left[\begin{array}{l} a+r, a'+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, g+2r; \end{array} x, y \right], \tag{3.9}$$

$$\kappa_3\left[\begin{matrix} a, a', b, b', c, c', d, d'; \\ e, f, g; \end{matrix} \quad x, y\right] \tag{3.10}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (b')_r (c)_r (c')_r (d)_r (d')_r}{r!(g+r-1)_r (g)_{2r} (e)_{2r} (f)_{2r}} x^r y^r$$
$$\times\ \kappa_2\left[\begin{matrix} a+r, a'+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, g+2r, g+2r; \end{matrix} \quad x, y\right],$$

$$\kappa_4\left[\begin{matrix} a, b, b, c, c', d, d'; \\ e, e', f, f', g, g'; \end{matrix} \quad x, y\right] \tag{3.11}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r} (b)_r (c)_r (c')_r (d)_r (d')_r}{r!(e)_r (e')_r (f)_r (f')_r (g)_r (g'_r)} x^r y^r$$
$$\times\ \kappa_8\left[\begin{matrix} a+2r, b+r, c+r, c'+r, d+r, d'+r; \\ e+r, e'+r, f+r, f'+r, g+r, g'+r; \end{matrix} \quad x, y\right],$$

$$\kappa_8\left[\begin{matrix} a, b, c, c', d, d'; \\ e, e', f, f', g, g'; \end{matrix} \quad x, y\right] \tag{3.12}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r} (b')_r (c)_r (c')_r (d)_r (d')_r}{r!(e)_r (e')_r (f)_r (f')_r (g)_r (g'_r)} x^r y^r$$
$$\times\ \kappa_4\left[\begin{matrix} a+2r, b+r, c+r, c'+r, d+r, d'+r; \\ e+r, e'+r, f+r, f'+r, g+r, g'+r; \end{matrix} \quad x, y\right],$$

$$\kappa_7\left[\begin{matrix} a, b, b', c, c', d, d'; \\ e, f, g; \end{matrix} \quad x, y\right] \tag{3.13}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (b')_r (c)_r (c')_r (d)_r (d')_r}{r!(e)_{2r} (f)_{2r} (g)_{2r}} x^r y^r$$
$$\times\ \kappa_3\left[\begin{matrix} a+r, a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, g+2r; \end{matrix} \quad x, y\right],$$

$$\kappa_3\left[\begin{matrix} a, a, b, b', c, c', d, d'; \\ e, f, g; \end{matrix} \quad x, y\right] \tag{3.14}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_r (b')_r (c)_r (c')_r (d)_r (d')_r}{r!(e)_{2r} (f)_{2r} (g)_{2r}} x^r y^r$$
$$\times\ \kappa_7\left[\begin{matrix} a+r, b+r, b'+r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+2r, g+2r; \end{matrix} \quad x, y\right],$$

$$\kappa_5\left[\begin{matrix} a, b, b, c, c', d, d'; \\ e, f, f'g, g'; \end{matrix} \quad x, y\right] \tag{3.15}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_{2r} (c)_r (c')_r (d)_r (d')_r}{r!(e)_{2r} (f)_r (f')_r (g)_r (g')_r} x^r y^r$$

$$\times\ \kappa_9\left[\begin{array}{ll} a+r, b+2r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+r, f'+r, g+r, g'+r; & x, y \end{array}\right],$$

(3.16)
$$\kappa_9\left[\begin{array}{ll} a, b, c, c', d, d'; & \\ e, f, f'g, g'; & x, y \end{array}\right]$$
$$= \sum_{r=0}^{\infty}\frac{(a)_{2r}(b)_r(b)_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_r(f')_r(g)_r(g')_r}x^r y^r$$
$$\times\ \kappa_5\left[\begin{array}{ll} a+2r, b+r, b+r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+r, f'+r, g+r, g'+r; & x, y \end{array}\right],$$

(3.17)
$$\kappa_{10}\left[\begin{array}{ll} a, b, c, c', d, d'; & \\ e, f, g, g'; & x, y \end{array}\right]$$
$$= \sum_{r=0}^{\infty}\frac{(a)_{2r}(b)_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_r(g')_r}x^r y^r$$
$$\times\ \kappa_6\left[\begin{array}{ll} a+2r, b+r, b+r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+2r, g+r, g'+r; & x, y \end{array}\right],$$

(3.18)
$$\kappa_6\left[\begin{array}{ll} a, b, b, c, c', d, d'; & \\ e, f, g, g'; & x, y \end{array}\right]$$
$$= \sum_{r=0}^{\infty}\frac{(-1)^r(a)_{2r}(b)_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_r(g')_r}x^r y^r$$
$$\times\ \kappa_{10}\left[\begin{array}{ll} a+2r, b+r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+2r, g+r, g'+r; & x, y \end{array}\right],$$

(3.19)
$$\kappa_7\left[\begin{array}{ll} a, b, b, c, c', d, d'; & \\ e, f, g; & x, y \end{array}\right]$$
$$= \sum_{r=0}^{\infty}\frac{(-1)^r(a)_{2r}(b)_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_{2r}}x^r y^r$$
$$\times\ \kappa_{11}\left[\begin{array}{ll} a+2r, b+r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+2r, g+2r; & x, y \end{array}\right],$$

(3.20)
$$\kappa_{11}\left[\begin{array}{ll} a, b, c, c', d, d'; & \\ e, f, g; & x, y \end{array}\right]$$
$$= \sum_{r=0}^{\infty}\frac{(a)_{2r}(b)_r(c)_r(c')_r(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_{2r}}x^r y^r$$
$$\times\ \kappa_{11}\left[\begin{array}{ll} a+2r, b+r, b+r, c+r, c'+r, d+r, d'+r; & \\ e+2r, f+2r, g+2r; & x, y \end{array}\right],$$

(3.21)
$$\kappa_{13}\left[\begin{array}{ll} a, b, c, d, d'; & \\ e, f, f', g, g'; & x, y \end{array}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_r(d)_r(d')_r}{r!(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_9 \left[\begin{array}{l} a+2r, b+2r, c+r, c'+r, d+r, d'+r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{array} \; x, y \right],$$

$$\kappa_9 \left[\begin{array}{l} a, b, c, c, d, d'; \\ e, f, f', g, g'; \end{array} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_r(d)_r(d')_r}{r!(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_{13} \left[\begin{array}{l} a+2r, b+2r, c+r, d+r, d'+r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{array} \; x, y \right], \tag{3.22}$$

$$\kappa_{13} \left[\begin{array}{l} a, b, c, d, d'; \\ e, f, f', g, g'; \end{array} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e+r-1)_r(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_{12} \left[\begin{array}{l} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, e+2r, f+r, f'+r, g+r, g'+r; \end{array} \; x, y \right], \tag{3.23}$$

$$\kappa_{12} \left[\begin{array}{l} a, b, c, d, d'; \\ e, e, f, f', g, g'; \end{array} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e)_{2r}(e)_r(f)_r(f')_r(g)_r(g')_r} x^r y^r \times \kappa_{12} \left[\begin{array}{l} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{array} \; x, y \right], \tag{3.24}$$

$$\kappa_{13} \left[\begin{array}{l} a, b, c, d, d'; \\ e, f, f, g, g'; \end{array} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e)_{2r}(f)_r(f)_{2r}(g)_r(g')_r} x^r y^r \times \kappa_{14} \left[\begin{array}{l} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, f+2r, g+r, g'+r; \end{array} \; x, y \right], \tag{3.25}$$

$$\kappa_{14} \left[\begin{array}{l} a, b, c, d, d'; \\ e, f, g, g'; \end{array} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e)_{2r}(f+r-1)_r(f)_{2r}(g)_r(g')_r} x^r y^r \times \kappa_{13} \left[\begin{array}{l} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, f+r, f+r, g+r, g'+r; \end{array} \; x, y \right], \tag{3.26}$$

$$\kappa_{15}\left[\begin{matrix} a,b,c,d,d'; \\ e,f,g; \end{matrix} \quad x,y\right] \tag{3.27}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e)_{2r}(g+r-1)_r(f)_{2r}(g)_{2r}} x^r y^r$$
$$\times\ \kappa_{14}\left[\begin{matrix} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, f+2r, g+r, g'+r; \end{matrix} \quad x,y\right],$$

$$\kappa_{14}\left[\begin{matrix} a,b,c,d,d'; \\ e,f,g,g; \end{matrix} \quad x,y\right] \tag{3.28}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_{2r}(d)_r(d')_r}{r!(e)_{2r}(f)_{2r}(g)_{2r}(g)_r} x^r y^r$$
$$\times\ \kappa_{15}\left[\begin{matrix} a+2r, b+2r, c+2r, d+r, d'+r; \\ e+2r, f+2r, g+2r; \end{matrix} \quad x,y\right],$$

$$\kappa_{16}\left[\begin{matrix} a,b,c,d; \\ e,e,f,f',g,g'; \end{matrix} \quad x,y\right] \tag{3.29}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_{2r}(d)_{2r}}{r!(e)_r(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r$$
$$\times\ \kappa_{17}\left[\begin{matrix} a+2r, b+2r, c+2r, d+2r; \\ e+2r, f+r, f'+r, g+r, g'+r; \end{matrix} \quad x,y\right],$$

$$\kappa_{17}\left[\begin{matrix} a,b,c,d; \\ e,f,f',g,g'; \end{matrix} \quad x,y\right] \tag{3.30}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_{2r}(d)_{2r}}{r!(e+r-1)_r(e)_{2r}(f)_r(f')_r(g)_r(g')_r} x^r y^r$$
$$\times\ \kappa_{16}\left[\begin{matrix} a,b,c,d; \\ e,e,f,f',g,g'; \end{matrix} \quad x,y\right],$$

$$\kappa_{18}\left[\begin{matrix} a,b,c,d; \\ e,f,g,g'; \end{matrix} \quad x,y\right] \tag{3.31}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_{2r}(d)_{2r}}{r!(f+r-1)_r(e)_{2r}(f)_{2r}(g)_r(g')_r} x^r y^r$$
$$\times\ \kappa_{17}\left[\begin{matrix} a+2r, b+2r, c+2r, d+2r; \\ e+2r, f+r, f+r, g+r, g'+r; \end{matrix} \quad x,y\right],$$

$$\kappa_{17}\left[\begin{matrix} a,b,c,d; \\ e,f,f,g,g'; \end{matrix} \quad x,y\right] \tag{3.32}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}(c)_{2r}(d)_{2r}}{r!(e)_{2r}(f)_{2r}(f)_r(g)_r(g')_r} x^r y^r$$

$$\times\ \kappa_{18}\left[\begin{array}{ll} a+2r, b+2r, c+2r, d+2r; & \\ e+2r, f+2r, g+r, g'+r; & x, y \end{array}\right].$$

Conclusion

We conclude our study by mentioning that, whenever a Appell's type functions $\kappa_i, i = 1, 2, \ldots, 18$ reduces to the Appell's type functions and other related hypergeometric functions, the results become relatively more important from the application viewpoint. Most of the special functions of mathematical physics and engineering, such as the Jacobi and Laguerre polynomials, can be expressed in terms of the hypergeometric function and other related hypergeometric functions. Therefore, the numerous differential formulas for generalized Appell's type functions $\kappa_i, i = 1, 2, \ldots, 18$ by considering the product of two ${}_4F_3$ functions are capable of playing important roles in the theory of special functions of applied mathematics and mathematical physics and we can easily derived many expansions for $\kappa_i, i = 1, 2, \ldots, 18$.

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