Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 623–642
https://doi.org/10.4134/CKMS.c230303
pISSN: 1225-1763 / eISSN: 2234-3024

ON ϕ-(n,d) RINGS AND ϕ-n-COHERENT RINGS

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Abstract. This paper introduces and studies a generalization of (n,d)-rings introduced and studied by Costa in 1994 to rings with prime nilradical. Among other things, we establish that the ϕ-von Neumann regular rings are exactly either ϕ-$(0,0)$ or ϕ-$(1,0)$ rings and that the ϕ-Prüfer rings which are strongly ϕ-rings are the ϕ-$(1,1)$ rings. We then introduce a new class of rings generalizing the class of n-coherent rings to characterize the nonnil-coherent rings introduced and studied by Bacem and Benhissi.

1. Introduction

All rings considered in this paper are assumed to be commutative with nonzero identity and prime nilradical. We use $\mathrm{Nil}(R)$ to denote the set of nilpotent elements of R, and $Z(R)$ to denote the set of zero-divisors of R. A ring with $\mathrm{Nil}(R)$ that is divided prime (i.e., $\mathrm{Nil}(R) \subset xR$ for every $x \in R \setminus \mathrm{Nil}(R)$) is called a ϕ-ring. Let $\mathcal{H}$ be the set of all ϕ-rings. A ring R is called a strongly ϕ-ring if $R \in \mathcal{H}$ and $Z(R) = \mathrm{Nil}(R)$. Let R be a ring and M be an R-module, we define

$$\phi\text{-}\mathrm{tor}(M) = \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus \mathrm{Nil}(R)\}.$$

If $\phi\text{-}\mathrm{tor}(M) = M$, then M is called a ϕ-torsion module, and if $\phi\text{-}\mathrm{tor}(M) = 0$, then M is called a ϕ-torsion free module. An ideal I of R is said to be nonnil if $I \not\subseteq \mathrm{Nil}(R)$. An R-module M is said to be ϕ-divisible if $M = sM$ for every $s \in R \setminus \mathrm{Nil}(R)$. An R-module M is said to be ϕ-uniformly torsion (ϕ-u-torsion for short) if $sM = 0$ for some $s \in R \setminus \mathrm{Nil}(R)$ [12, Definition 2.2].

Let R be a ring and n be a nonnegative integer. According to Costa [9], an R-module M is said to be n-presented if there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ such that each F_i is a finitely generated free R-module, equivalently each F_i is a finitely generated projective R-module.

Received November 16, 2023; Accepted February 29, 2024.
2020 *Mathematics Subject Classification.* 13A02, 13A15.
Key words and phrases. Nonnil-coherent ring, ϕ-Noetherian ring, ϕ-n-presented module, nonnil-FP-injective module, ϕ-(n,d)-injective modules, ϕ-(n,d)-flat modules, ϕ-(n,d)-ring, ϕ-weak-(n,d)-ring, ϕ-n-coherent ring, ϕ-n-von Neumann regular ring.

If M is a ϕ-torsion R-module that is n-presented, then M is called a ϕ-n-presented module. A finite n-presentation of a ϕ-torsion R-module is said to be a ϕ-n-presentation. Obviously, every finitely generated projective module is n-presented for every n. A module is 0-presented (resp., 1-presented) if and only if it is finitely generated (resp., finitely presented), and every m-presented module is n-presented for any $n \leq m$. A ring R is called n-coherent if every n-presented R-module is $(n+1)$-presented. It is easy to see that R is 0-coherent (resp., 1-coherent) if and only if it is Noetherian (resp., coherent), and every n-coherent ring is m-coherent for any $m \geq n$. The n-coherent ring is further studied in detail in [10, 11]. Costa introduced a doubly filtered set of classes of rings to categorize the structure of non-Noetherian rings for nonnegative integers n and d. We say that a ring R is an (n, d)-ring if $\mathrm{pd}_R(M) \leq d$ for every n-presented R-module M (as usual, $\mathrm{pd}_R(M)$ denotes the projective dimension of M as an R-module). An integral domain with this property is called an (n, d)-domain. For example, the $(n, 0)$-domains are the fields, the $(0, 1)$-domains are the Dedekind domains, and the $(1, 1)$-domains are the Prüfer domains [9]. The (n, d)-ring is further studied in detail in [16, 19, 21–23]. We call a commutative ring an n-von Neumann regular ring if it is an $(n, 0)$-ring. Thus, the 1-von Neumann regular rings are exactly the von Neumann regular rings [9, Theorem 1.3].

In 2004, D. Zhou [30] introduced and studied a new class of modules with two parameters $n, d \in \mathbb{N}$, the set of nonnegative integers: an R-module N is said to be (n, d)-injective (resp., (n, d)-flat) if $\mathrm{Ext}_R^{d+1}(M, N) = 0$ (resp., $\mathrm{Tor}^R_{d+1}(M, N) = 0$ for $n \geq 1$) for each n-presented R-module M. In particular, the $(0, 0)$-injective modules are injective, the $(1, 0)$-injective modules are FP-injective (i.e., modules N in which we have $\mathrm{Ext}_R^1(M, N) = 0$ for every finitely presented R-module M), more generally, an R-module M is $(0, d)$-injective if the injective dimension of M is at most d. An R-module M is $(1, 0)$-flat if it is flat, and M is $(1, d)$-flat if the flat dimension of M is at most d. A ring R is called a weak-(n, d)-ring with $n \geq 1$ if each n-presented module has a flat dimension at most d. In particular, the weak-$(1, 0)$-rings are von Neumann regular rings. D. Zhou established that a ring R is n-coherent if and only if every $(n + 1, 0)$-injective module is $(n, 0)$-injective, and if $n \geq 1$, then R is n-coherent if and only if every $(n + 1, 0)$-flat module is $(n, 0)$-flat [30, Theorem 3.4].

In 1996, J. Chen and N. Ding [8] introduced a generalization of flat modules and injective modules by a nonzero positive integer parameter. An R-module N is said to be n-flat (with $n \geq 1$) (resp., n-FP-injective) if $\mathrm{Tor}_n^R(M, N) = 0$ (resp., $\mathrm{Ext}_R^n(M, N) = 0$) for every n-presented R-module M. In other words, the n-flat (resp., n-FP-injective) modules are $(n, n-1)$-flat (resp., $(n, n-1)$-injective). They characterized the n-coherent rings by the n-flat modules and the n-FP-injective modules (see [8, Theorem 3.1]).

In [2], D. F. Anderson and A. Badawi introduced a class of ϕ-rings called ϕ-Prüfer. A ϕ-ring R is said to be ϕ-Prüfer if $R/\operatorname{Nil}(R)$ is a Prüfer domain [2, Theorem 2.6]. All ϕ-Prüfer rings are Prüfer [2, Theorem 2.14], if additionally $Z(R) = \operatorname{Nil}(R)$, then every Prüfer ring is ϕ-Prüfer [2, Theorem 2.16]. In [29], G. Tang, F. Wang, and W. Zhao introduced a class of ϕ-rings which are called ϕ-von Neumann regular rings. An R-module M is said to be ϕ-flat if for every monomorphism $f : A \to B$ with $\operatorname{Coker}(f)$ ϕ-torsion, $f \otimes 1 : A \otimes_R M \to B \otimes_R M$ is an R-monomorphism [29, Definition 3.1]. An R-module M is ϕ-flat if and only if $M_{\mathfrak{p}}$ is ϕ-flat for every prime ideal $\mathfrak{p}$ of R, if and only if $M_{\mathfrak{m}}$ is ϕ-flat for every maximal ideal $\mathfrak{m}$ of R [29, Theorem 3.5]. A ϕ-ring R is said to be a ϕ-von Neumann regular ring if all R-modules are ϕ-flat, which is equivalent to saying that $R/\operatorname{Nil}(R)$ is a von Neumann regular ring [29, Theorem 4.1].

Recall from [4] that a ϕ-ring R is said to be nonnil-Noetherian if $R/\operatorname{Nil}(R)$ is a Noetherian domain, which is equivalent to saying that every nonnil ideal of R is finitely generated. Note that this notion coincides with the notion of ϕ-Noetherian rings in the work of the authors of [5].

In [3], K. Bacem and B. Ali introduced two new classes of ϕ-rings: a ϕ-ring R is said to be ϕ-coherent if $R/\operatorname{Nil}(R)$ is a coherent domain [3, Corollary 3.1]; a ϕ-ring R is said to be nonnil-coherent if every finitely generated nonnil ideal of R is finitely presented, which is equivalent to saying that R is ϕ-coherent and $(0 : r)$ is a finitely generated ideal of R for every $r \in R \setminus \operatorname{Nil}(R)$ [24, Proposition 1.3]. Following Y. El Haddaoui, H. Kim, and N. Mahdou [13], a submodule N of an R-module M is said to be a ϕ-submodule if M/N is a ϕ-torsion module [13, Definition 2.1]. For $R \in \mathcal{H}$, an R-module M is said to be nonnil-coherent if M is finitely generated and every finitely generated ϕ-submodule of M is finitely presented [13, Definition 2.2]. It is easy to see that every coherent module over a ϕ-ring is nonnil-coherent. Next they established in [13, Theorem 2.6] the analog of the well-known behavior of the relation between the coherent rings and the finitely generated submodules of a finitely generated free module.

Y. El Haddaoui and N. Mahdou [12] introduced and studied the ϕ-(weak) global dimension of rings with prime nilradical. An R-module P is said to be ϕ-u-projective if $\operatorname{Ext}^1_R(P, N) = 0$ for any ϕ-u-torsion R-module N [12, Definition 3.1]. The ϕ-projective dimension of M over R, denoted by $\phi\text{-}\operatorname{pd}_R M$, is said to be at most n (where $n \in \mathbb{N}^*$) if either $M = 0$ or M is not a ϕ-u-projective module which satisfies $\operatorname{Ext}^{n+1}_R(M, N) = 0$ for every ϕ-u-torsion module N. In addition, if n is the least such nonnegative integer, then we set $\phi\text{-}\operatorname{pd}_R M = n$. If no such n exists, we set $\phi\text{-}\operatorname{pd}_R M = \infty$ [12, Definition 3.2]. For a ring R with $Z(R) = \operatorname{Nil}(R)$, define

$$\phi\text{-}\operatorname{gl.dim}(R) = \sup\left\{\phi\text{-}\operatorname{pd}_R R/I \mid I \text{ is a nonnil ideal of } R\right\},$$

which is called the ϕ-global dimension of R [12, Definition 4.1]. Similarly, the ϕ-flat dimension of an R-module M, denoted by $\phi\text{-}\operatorname{fd}_R M$, is said to be at most n (where $n \in \mathbb{N}^*$) if either $M = 0$ or M is not ϕ-flat which satisfies $\operatorname{Tor}^R_{n+1}(M, N) = 0$ for every ϕ-u-torsion module N. In addition, if n is at least one such nonnegative integer, then we set $\phi\text{-fd}_R\, M = n$. If there is no such n, we set $\phi\text{-fd}_R\, M = \infty$ [12, Definition 5.7]. Let R be a ring. Define for a ring R with $Z(R) = Nil(R)$

$$\begin{aligned}
\phi\text{- w. gl. dim}(R) &= \sup\{\phi\text{- fd}_R\, M \mid M \text{ is } \phi\text{-torsion}\} \\
&= \sup\{\phi\text{- fd}_R\, M \mid M \text{ is } \phi\text{-u-torsion}\} \\
&= \sup\{\phi\text{- fd}_R\, M \mid M \text{ is finitely presented } \phi\text{-torsion}\} \\
&= \sup\{\phi\text{- fd}_R\, M \mid M \text{ is finitely presented } \phi\text{-u-torsion}\} \\
&= \sup\{\phi\text{- fd}_R\, R/I \mid I \text{ is a nonnil ideal of } R\} \\
&= \sup\{\phi\text{- fd}_R\, R/I \mid I \text{ is a finitely generated nonnil ideal of } R\},
\end{aligned}$$

which is called the ϕ-weak global dimension of R [12, Definition 5.10]. If $R \in \mathcal{H}$, then R is a ϕ-von Neumann regular ring if and only if $\phi\text{- w. gl. dim}(R) = 0$ [12, Theorem 5.29], which is equivalent to saying that $\phi\text{- gl. dim}(R) = 0$ [12, Corollary 5.33]. A strongly ϕ-ring is ϕ-Prüfer if and only if $\phi\text{-w. gl. dim}(R) \leq 1$ [12, Corollary 5.27] if and only if every finitely generated nonnil ideal of R is ϕ-u-projective [12, Theorem 5.41].

Our paper consists of three sections, including the introduction. In Section 2 we introduce ϕ-(n, d)-rings, which are generalizations of the (n, d)-rings (where $n, d \geq 0$ are integers) introduced and studied by D. L. Costa [9]. An R-module N is said to be ϕ-(n, d)-injective or nonnil (n, d)-injective if $\text{Ext}_R^{d+1}(R/I, N) = 0$ for every nonnil ideal I of R such that R/I is a ϕ-n-presented module (see Definition 2). An R-module M is said to be ϕ-(n, d)-flat (with $n \in \mathbb{N}^*$, the set of positive integers) if $\text{Tor}_{d+1}^R(R/I, N) = 0$ for every ϕ-n-presented module R/I, where I is a nonnil ideal of R. A ring R is said to be a ϕ-(n, d)-ring if every ϕ-n-presented module M has a ϕ-projective dimension at most d. We establish in Theorem 2.22 that the ϕ-von Neumann regular rings are exactly either ϕ-$(0, 0)$ or ϕ-$(1, 0)$ rings and that the ϕ-Prüfer rings which are strongly ϕ-rings are the ϕ-$(1, 1)$ rings. In Section 3, we define a generalization of n-coherent rings. A ring R is said to be ϕ-n-coherent if all ϕ-n-presented R-modules are ϕ-$(n+1)$-presented. We give several equivalent conditions for a ring to be ϕ-n-coherent. We show that there are many similarities between coherent rings and ϕ-n-coherent rings. For example, a ring R is ϕ-n-coherent if and only if every direct product of R is a ϕ-n-flat R-module, if and only if every direct product of ϕ-n-flat R-modules is ϕ-n-flat, if and only if every direct limit of ϕ-n-FP-injective R-modules (which are ϕ-$(n, n-1)$-injectives) is ϕ-n-FP-injective (see Theorem 3.10).

For any undefined terminology and notation, the reader may refer to [14, 26, 27].

2. ϕ-(n, d)-rings

In this section, we introduce and study a generalization of (n, d)-rings (where $n, d \geq 0$ are integers) introduced and studied by D. L. Costa [9].

Definition 1. Let R be a ring. An R-module M is said to be n-presented if M has an n-finite presentation. In addition, if M is a ϕ-torsion R-module, then M is said to be ϕ-n-presented and the n-finite presentation is called a ϕ-n-presentation of M.

Remark 2.1. If $m \leq n$ are nonnegative integers, then every ϕ-n-presented module is ϕ-m-presented.

Proposition 2.2. *Let R be a ring and M be an R-module. Then*

(1) *M is ϕ-0-presented if and only if M is a finitely generated ϕ-torsion R-module.*
(2) *M is ϕ-1-presented if and only if M is a finitely presented ϕ-torsion R-module.*

Proof. This is straightforward. □

Definition 2. Let R be a ring and $n, d \in \mathbb{N}$. An R-module N is said to be ϕ-(n,d)-injective or nonnil (n,d)-injective if $\mathrm{Ext}_R^{d+1}(R/I, N) = 0$ for every nonnil ideal I such that R/I is a ϕ-n-presented R-module.

Definition 3. Let R be a ring. An R-module N is called ϕ-FP-injective if $\mathrm{Ext}_R^1(R/I, N) = 0$ for every finitely generated nonnil ideal of R.

By [13, Theorem 2.6], a ϕ-ring R is nonnil-coherent if and only if every finitely generated ϕ-submodule of a finitely presented module is also finitely presented. From [24, Definition 1.7], an R-module N is said to be nonnil-FP-injective if $\mathrm{Ext}_R^1(M, N) = 0$ for every finitely presented ϕ-torsion module M. Next, we prove that every ϕ-FP-injective module over a nonnil-coherent ring is nonnil-FP-injective

Proposition 2.3. *If R is a nonnil-coherent ring, then every ϕ-FP-injective module is nonnil-FP-injective.*

Proof. Let N be a ϕ-FP-injective module. Then $\mathrm{Ext}_R^1(R/I, N) = 0$ for every finitely generated nonnil ideal of I of R. We claim that $\mathrm{Ext}_R^1(F, N) = 0$ for every finitely presented ϕ-torsion R-module F. Let F be a finitely presented ϕ-torsion module. We use induction on the number of generators of F. Assume that F is a finitely presented ϕ-torsion module on m generators, and let F' be the submodule generated by one of these generators. Since R is nonnil-coherent, both F' and F/F' are finitely presented ϕ-torsions on less than m generators, so we get an exact sequence $\mathrm{Ext}_R^1(F/F', N) \to \mathrm{Ext}_R^1(F, N) \to \mathrm{Ext}_R^1(F', N)$, where both end terms are zero by induction. Thus $\mathrm{Ext}_R^1(F, N) = 0$. Hence N is nonnil-FP-injective. □

According to [28], an R-module E is said to be nonnil-injective if

$$\mathrm{Ext}_R^1(R/I, E) = 0$$

for every nonnil ideal I of R. Recall from [12] that the ϕ-injective dimension of M over R, denoted by $\phi\text{-}\mathrm{id}_R\, M$, is said to be at most $n \geq 1$ (where $n \in \mathbb{N}$)

if either $M = 0$ or $M \neq 0$ which is not nonnil-injective and which satisfies $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ for every nonnil ideal I of R. If n is the least nonnegative integer for which $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ for every nonnil ideal I of R, then we set $\phi\text{-id}_R M = n$. If there is no such n, we set $\phi\text{-id}_R M = \infty$ [12, Definition 2.5], and it is easy to see that an R-module M is of ϕ-injective dimension zero if and only if it is nonnil-injective. We also have that for a ring R with $Z(R) = \mathrm{Nil}(R)$,

$$\phi\text{-}\operatorname{gl.dim}(R) = \sup\{\phi\text{-}\operatorname{id}_R N \mid N \text{ is a } \phi\text{-}u\text{-torsion } R\text{-module}\}.$$

Proposition 2.4. *Let N be an R-module. Then the following statements hold:*

(1) *N is a ϕ-$(0,0)$-injective module if and only if N is a nonnil-injective module.*
(2) *If $d \geq 1$ and N is not nonnil-injective, then N is a ϕ-$(0,d)$-injective module if and only if $\phi\text{-}\operatorname{id}_R N \leq d$.*
(3) *N is a ϕ-$(1,0)$-injective module if and only if N is a ϕ-FP-injective module.*

Proof. (1) N is a ϕ-$(0,0)$-injective module if and only if $\mathrm{Ext}_R^1(R/I, N) = 0$ for every nonnil ideal I of R, if and only if N is a nonnil-injective module. (2) This follows from [12, Theorem 2.6]. (3) This follows from Definition 3. □

Definition 4. Let R be a ring and let $(n, d) \in \mathbb{N}^* \times \mathbb{N}$. An R-module M is said to be ϕ-(n,d)-flat if $\mathrm{Tor}_{d+1}^R(R/I, N) = 0$ for every nonnil ideal I of R such that R/I is a ϕ-n-presented module.

Proposition 2.5. *Let M be an R-module. The following statements hold:*

(1) *M is a ϕ-$(1,0)$-flat module if and only if M is a ϕ-flat module.*
(2) *If $d \geq 1$ and M is not ϕ-flat, then M is a ϕ-$(1,d)$-flat module if and only if $\phi\text{-}fd_R M \leq d$.*

Proof. (1) M is a ϕ-$(1,0)$-flat module if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for every finitely generated nonnil ideal I of R, if and only if M is a ϕ-flat module by [29, Theorem 3.2].

(2) This follows from [12, Theorem 5.19]. □

Proposition 2.6. *Let m, n and d be nonnegative integers such that $m \leq n$. Then:*

(1) *Every ϕ-(m,d)-injective module is ϕ-(n,d)-injective.*
(2) *If $m \geq 1$, then every ϕ-(m,d)-flat module is ϕ-(n,d)-flat.*

Proof. This follows immediately from Remark 2.1 and Definitions 2 and 4. □

Next, we give some properties related to ϕ-(n,d)-rings, ϕ-(n,d)-injective modules, and ϕ-(n,d)-flat modules.

Theorem 2.7. *Let $\{N_i\}_{i\in\Gamma}$ be a family of R-modules. Then $\prod_{i\in\Gamma} N_i$ is a ϕ-(n,d)-injective module if and only if each N_i is ϕ-(n,d)-injective.*

Proof. Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module. From $\mathrm{Ext}_R^{d+1}(R/I, \prod_{i\in\Gamma} N_i) \cong \prod_{i\in\Gamma} \mathrm{Ext}_R^{d+1}(R/I, N_i)$, we get that $\prod_{i\in\Gamma} N_i$ is a ϕ-(n,d)-injective module if and only if each N_i is ϕ-(n,d)-injective. □

Theorem 2.8. *Let $\{M_i\}_{i\in\Gamma}$ be a family of R-modules and $n \geq 1$. Then $\bigoplus_{i\in\Gamma} M_i$ is a ϕ-(n,d)-flat module if and only if each M_i is ϕ-(n,d)-flat.*

Proof. Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module. Since

$$\mathrm{Tor}_{d+1}^R(R/I, \bigoplus_{i\in\Gamma} M_i) \cong \bigoplus_{i\in\Gamma} \mathrm{Tor}_{d+1}^R(R/I, M_i),$$

we get that $\bigoplus_{i\in\Gamma} M_i$ is a ϕ-(n,d)-flat module if and only if each M_i is ϕ-(n,d)-flat. □

In this paper, for a ϕ-n-presented module M with a ϕ-n-presentation

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

we set $K_i := \ker(F_i \longrightarrow F_{i-1})$ for all $0 \leq i \leq n$ and $F_{-1} := M$.

The following result characterizes the ϕ-(n,d)-injective modules.

Theorem 2.9. *The following statements are equivalent for an R-module N such that $n \geq d+1$.*

(1) *N is a ϕ-(n,d)-injective module.*

(2) *For every nonnil ideal I such that R/I is a ϕ-n-presented module with a ϕ-n-presentation*

$$F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0,$$

we get $\mathrm{Ext}_R^1(K_{d-1}, N) = 0$.

(3) *For every nonnil ideal I such that R/I is a ϕ-n-presented module with a ϕ-n-presentation*

$$F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0,$$

and every R-homomorphism $f : K_d \longrightarrow N$, f can be extended to F_d.

Proof. (1) ⇒ (2) Assume that N is a ϕ-(n,d)-injective module. Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module with a ϕ-n-presentation $F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0$. Since $n \geq d+1$, it follows that R/I is ϕ-d-presented, and so we have $\mathrm{Ext}_R^{d+1}(R/I, N) \cong \mathrm{Ext}_R^1(K_{d-1}, N) = 0$.

(2) ⇒ (3) Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module with a ϕ-n-presentation $F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0$. Assume that $\mathrm{Ext}_R^1(K_{d-1}, N) = 0$ and let $f : K_d \longrightarrow N$ be an R-homomorphism. Then we have the following exact sequence $0 \to K_d \to F_d \to K_{d-1} \to 0$, which induces the exact sequence $0 \to \mathrm{Hom}_R(K_{d-1}, N) \to \mathrm{Hom}_R(F_d, N) \to \mathrm{Hom}_R(K_d, N) \to 0$. So f can be extended to F_d.

(3) ⇒ (1) Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module with a ϕ-n-presentation $F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0$. By

hypothesis, we have the exact sequence $\mathrm{Hom}_R(F_d, N) \to \mathrm{Hom}_R(K_d, N) \to 0$. From the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Hom}_R(F_d,N) & \longrightarrow & \mathrm{Hom}_R(K_d,N) & \longrightarrow & \mathrm{Ext}^1_R(K_{d-1},N) & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & \\ \mathrm{Hom}_R(F_d,N) & \longrightarrow & \mathrm{Hom}_R(K_d,N) & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

we get $\mathrm{Ext}^1_R(K_{d-1}, N) = 0$. In addition, $\mathrm{Ext}^{d+1}_R(R/I, N) \cong \mathrm{Ext}^1_R(K_{d-1}, N) = 0$, since $n \geq d+1$. So N is a ϕ-(n, d)-injective module. □

The following result characterizes the ϕ-(n, d)-flat modules.

Theorem 2.10. *The following statements are equivalent for an R-module N such that $n \geq d + 1$.*

(1) *N is a ϕ-(n, d)-flat modules.*
(2) *For every nonnil ideal I of R such that R/I is a ϕ-n-presented module with a ϕ-n-presentation*

$$F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0,$$

we get $\mathrm{Tor}^R_1(K_{d-1}, N) = 0$.
(3) *For every nonnil ideal I of R such that R/I is a ϕ-n-presented module with a ϕ-n-presentation*

$$F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0,$$

the sequence $0 \to N \otimes_R K_d \to N \otimes_R F_d$ *is exact.*

Proof. The proof is similar to the proof of Theorem 2.9. □

According to [25], an R-module N is an injective cogenerator if for every nonzero R-module M, we have $\mathrm{Hom}_R(M, N) \neq 0$. In particular, $\mathbb{Q}/\mathbb{Z}$ is an example of an injective cogenerator abelian group. For an R-module M, we set $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Theorem 2.11. *Let $n \geq 1$ be an integer. An R-module N is ϕ-(n, d)-flat if and only if N^+ is ϕ-(n, d)-injective.*

Proof. This follows immediately from the following isomorphism:

$$\mathrm{Ext}^{d+1}_R(R/I, N^+) \cong \mathrm{Tor}^R_{d+1}(R/I, N)^+.$$

□

Corollary 2.12. *The following are equivalent for an R-module N.*

(1) *N is a ϕ-flat module.*
(2) *N^+ is a ϕ-FP-injective module.*
(3) *N^+ is a nonnil-injective module.*

Proof. (1) $\Leftrightarrow$ (2) This is straightforward by Propositions 2.4 and 2.5, and Theorem 2.11.

(1) $\Leftrightarrow$ (3) This follows from the isomorphism:

$$\mathrm{Tor}_1^R(R/I,N)^+ \cong \mathrm{Ext}_R^1(R/I,N^+)$$

and [29, Theorem 3.2]. □

Theorem 2.13. *If $n \geq d+1$, then every pure submodule of a ϕ-(n,d)-injective module is ϕ-(n,d)-injective. Also, every pure submodule of a ϕ-(n,d)-flat module is ϕ-(n,d)-flat.*

Proof. Assume that $n \geq d+1$ and let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module with a ϕ-n-finite presentation

$$F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0.$$

Since $n \geq d+1$, it follows that $K := K_{d-1}$ is a finitely presented R-module. Let X be a pure submodule of a ϕ-(n,d)-injective module N. Then the sequence $0 \to \mathrm{Hom}_R(K,X) \to \mathrm{Hom}_R(K,N) \to \mathrm{Hom}_R(K,N/X) \to 0$ is exact. Furthermore, we have $\mathrm{Ext}_R^{d+1}(R/I,N) \cong \mathrm{Ext}_R^1(K,N) = 0$, and so we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(K,N) & \longrightarrow & \mathrm{Hom}_R(K,N/X) & \longrightarrow & \mathrm{Ext}_R^1(K,X) & \longrightarrow & 0\ . \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{Hom}_R(K,N) & \longrightarrow & \mathrm{Hom}_R(K,N/X) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Thus $\mathrm{Ext}_R^{d+1}(R/I,X) \cong \mathrm{Ext}_R^1(K,X) = 0$. Hence X is a ϕ-(n,d)-injective module.

Now, let X be a pure submodule of a ϕ-(n,d)-flat module F. Since $0 \to X \to F \to F/X \to 0$ is pure exact, the induced exact sequence $0 \to (F/X)^+ \to F^+ \to X^+ \to 0$ is split by [26, Chapter I, Exercise 40]. Since F^+ is a ϕ-(n,d)-injective module by Theorem 2.11 and $F^+ \cong (F/X)^+ \oplus X^+$, it follows that X^+ is a ϕ-(n,d)-injective module by Theorem 2.7. Therefore, X is a ϕ-(n,d)-flat module by Theorem 2.11. □

Definition 5. A ring R is said to be ϕ-(n,d) if every ϕ-n-presented module has ϕ-projective dimension at most d.

If $n \geq 1$, then a ring R is said to be ϕ-weak-(n,d) if every ϕ-n-presented module has ϕ-flat dimension at most d.

Proposition 2.14. *If $n \leq n'$ and $d \leq d'$ are nonzero integers, then every ϕ-(n,d) ring (resp., ϕ-weak-(n,d) ring with $n \geq 1$) is a ϕ-(n',d') (resp., ϕ-weak-(n',d')) ring.*

Proof. This is straightforward. □

Remark 2.15. Recall that $\overline{\mathcal{H}}$ is the set of all ϕ-rings whose nilradical is not a maximal ideal. Recall also from [29, Theorem 4.1] that R is a ϕ-von Neumann regular ring if and only if $R \notin \overline{\mathcal{H}}$.

Theorem 2.16. *Let R be a ring. If R is a ϕ-(n,d) ring, then every ϕ-u-torsion R-module is ϕ-(n,d)-injective.*

Before proving Theorem 2.16, we establish Lemma 2.17.

Lemma 2.17. *Let $R \in \overline{\mathcal{H}}$ and I be a finitely generated nonnil ideal of R. Then R/I is ϕ-u-projective if and only if $I = R$.*

Proof. First, we establish that the ϕ-rings are connected. In fact, if there exists a nontrivial idempotent e in R, then $e(1-e) \in \mathrm{Nil}(R)$ implies that either $e \in \mathrm{Nil}(R)$ or $1-e \in \mathrm{Nil}(R)$. But if $e \in \mathrm{Nil}(R)$, then $e = 0$, which is impossible. Then $1 - e \in \mathrm{Nil}(R)$, and so $e \in U(R)$, which is also impossible. Then R is connected. On the other hand, we have from [12, Corollary 5.36] that R/I is a projective R-module, and so I is generated by an idempotent by [1, Exercise (10.24)]. Then R/I is ϕ-u-projective if and only if $I = R$. □

Proof of Theorem 2.16. We prove this result for the case where $Z(R) = \mathrm{Nil}(R)$. Assume that R is a ϕ-(n,d)-ring, and let N be a ϕ-u-torsion R-module. Then for every ϕ-n-presented module R/I, where I is a nonnil ideal of R, we have that ϕ-$\mathrm{pd}_R(R/I) \leq d$, and so $\mathrm{Ext}_R^{d+1}(R/I,N) = 0$ by [12, Theorem 3.10 and Remark 5.3(2)]. Therefore, N is a ϕ-(n,d)-injective module. Now, if $Z(R) \neq \mathrm{Nil}(R)$, then necessarily $R \in \overline{\mathcal{H}}$. Lemma 2.17 justifies that R/I is never a ϕ-u-projective R-module if we assume that I is a proper nonnil ideal of R. We repeat the same previous proof, and we are done. □

Theorem 2.18. *Let $n \geq 1$ be an integer. Then the following are equivalent for a ring R.*

1. *R is a ϕ-weak-(n,d) ring.*
2. *Every nonnil ideal I of R, R/I is ϕ-(n,d)-flat.*
3. *Every finitely generated nonnil ideal I of R, R/I is ϕ-(n,d)-flat.*

Proof. (1) ⇒ (2) Let M be a ϕ-n-presented module and I be a nonnil ideal of R. By hypothesis, we get that M has a ϕ-flat dimension at most d, and so $\mathrm{Tor}_{d+1}^R(R/I,M) = 0$. Therefore, R/I is ϕ-(n,d)-flat.

(2) ⇒ (3) ⇒ (1) These are obvious. □

Theorem 2.19. *If R is a ϕ-(n,d) ring, then R is ϕ-weak-(n,d). The converse holds if $n \geq d+1$.*

Proof. Assume that R is a ϕ-(n,d) ring. Then ϕ-$\mathrm{pd}_R\, M \leq d$ for every ϕ-n-presented R-module M, and so ϕ-$\mathrm{fd}_R\, M \leq d$. Therefore, R is ϕ-weak-(n,d).

Assume that $n \geq d+1$ and R is a ϕ-weak-(n,d) ring. Let M be a ϕ-n-presented module with a ϕ-n-finite presentation

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0.$$

Since $n \geq d+1$, it follows that $K := \ker(F_{d-1} \to F_{d-2})$ is finitely presented. Moreover $\operatorname{Tor}_1^R(K,N) \cong \operatorname{Tor}_{d+1}^R(M,N) = 0$ for every ϕ-torsion R-module N. So K is a ϕ-flat module, and so K is ϕ-u-projective by [12, Theorem 5.13]. Thus ϕ-$\operatorname{pd}_R M \leq d$, and so R is a ϕ-(n,d) ring. □

Theorem 2.20. *Let R be a ring with $Z(R) = \operatorname{Nil}(R)$. If R is a ϕ-$(n,d+1)$ ring, then every factor of a ϕ-u-torsion ϕ-(n,d)-injective module is ϕ-(n,d)-injective.*

Proof. Let E be a ϕ-u-torsion ϕ-(n,d)-injective module. We claim that E/N is a ϕ-(n,d)-injective module for every submodule N of E. First, note that N and E/N are ϕ-u-torsion modules. Using the exact sequence $0 \to N \to E \to E/N \to 0$, we get the following isomorphism:

$$\operatorname{Ext}_R^{d+2}(R/I,N) \cong \operatorname{Ext}_R^{d+1}(R/I,E/N)$$

for every ϕ-n-presented module R/I, where I is a nonnil ideal of R. So $\operatorname{Ext}_R^{d+1}(R/I,E/N) = 0$, since R is assumed to be a ϕ-$(n,d+1)$ ring. Therefore, E/N is a ϕ-(n,d)-injective module. □

Theorem 2.21. *Let R be a ring with $Z(R) = \operatorname{Nil}(R)$. If R is a ϕ-$(n,d+1)$ ring, then every submodule of a ϕ-torsion ϕ-(n,d)-flat module is ϕ-(n,d)-flat.*

Proof. The proof is similar to the proof of Theorem 2.20. □

In [12], a ϕ-ring R is said to be ϕ-hereditary if every nonnil ideal of R is ϕ-u-projective.

The following result gives some examples of ϕ-(n,d) rings for small nonnegative integers n,d.

Theorem 2.22. *Let $R \in \mathcal{H}$. Then*

(1) *R is a ϕ-$(0,0)$ ring if and only if R is a ϕ-von Neumann regular ring.*
(2) *R is a ϕ-$(0,1)$ ring if and only if R is a ϕ-hereditary ring.*
(3) *R is a ϕ-$(1,0)$ ring if and only if R is a ϕ-von Neumann regular ring.*
(4) *R is a ϕ-$(1,1)$ ring if and only if R is a ϕ-Prüfer ring with $Z(R) = \operatorname{Nil}(R)$.*

To prove Theorem 2.22, we need the following Lemma 2.23. Recall from [12, Definition 5.1] that a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

is said to be ϕ-pure exact if for every finitely presented ϕ-torsion module F, we get the following exact sequence $0 \to F \otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$. In particular, every pure exact sequence is ϕ-pure. A submodule A of B is said to be ϕ-pure if the exact sequence $0 \to A \to B \to B/A \to 0$ is ϕ-pure.

Lemma 2.23. *Every ϕ-ring R with ϕ-$\operatorname{w.gl.dim}(R) \leq 1$ is a strongly ϕ-ring.*

Proof. Assume that ϕ-$\operatorname{w.gl.dim}(R) \leq 1$ such that $\operatorname{Nil}(R)$ is not a maximal ideal. If $\operatorname{Nil}(R) \subsetneq Z(R)$, then there exists $s \in Z(R) \setminus \operatorname{Nil}(R)$. But R is a

ϕ-ring. Then R is a connected ring, and so $\frac{R}{\langle s\rangle}$ can not be a ϕ-flat R-module by [12, Theorem 5.13 and Corollary 5.36]. Then $\langle s\rangle$ is a ϕ-flat ideal. By [12, Theorem 5.4], the short exact sequence $0 \to (0 : s) \to R \to \langle s\rangle \to 0$ is ϕ-pure, which implies that the R-homomorphism given by $\varphi :\ (0 : s)\otimes_R \frac{R}{\langle s\rangle} \to \frac{R}{\langle s\rangle}$ is an R-monomorphism. But its kernel equals to $\frac{\langle s\rangle}{s(0:s)}$. Then $\langle s\rangle = s(0 : s)$, in particular, $s = rs$ for some $r \in (0 : s)$, and so $s = 0$, a contradiction. Consequently, we proved that $Z(R) = \mathrm{Nil}(R)$. □

Proof of Theorem 2.22. (1) R is a ϕ-$(0, 0)$ ring if and only if ϕ-gl. dim$(R) = 0$; if and only if R is a ϕ-von Neumann regular ring by [12, Corollary 5.33].

(2) It follows from Lemma 2.23 and [12, Proposition 5.25] that R is a ϕ-$(0, 1)$ ring if and only if ϕ-gl. dim$(R) \leq 1$; if and only if R is a ϕ-hereditary ring by [12, Theorem 4.3].

(3) Assume that R is a ϕ-$(1, 0)$ ring. If $R \in \overline{\mathcal{H}}$, then there exists a finitely generated proper nonnil ideal of R. By Lemma 2.17, R/I is never ϕ-u-projective. But R is a ϕ-$(1, 0)$ ring, then ϕ-pd$_R(R/I) = 0$, i.e., R/I is ϕ-u-projective, a contradiction. Therefore, R is a ϕ-von Neumann regular ring by Remark 2.15.

(4) Assume that R is a ϕ-$(1, 1)$. Then $Z(R) = \mathrm{Nil}(R)$ by Lemma 2.23. Let I be a finitely generated nonnil ideal of R. Then ϕ-pd$_R(R/I) \leq 1$, and so I is ϕ-u-projective. Therefore, R is a ϕ-Prüfer ring by [12, Theorem 5.41].

Conversely, assume that R is a ϕ-Prüfer ring, and let F be a finitely presented ϕ-torsion R-module. Then F is a factor of a finitely generated free R-module L by a finitely generated submodule of L, which is ϕ-u-projective by [12, Theorem 5.41], and so ϕ-pd$_R F \leq 1$. Therefore, R is a ϕ-$(1, 1)$ ring. □

3. On ϕ-n-coherent rings

In this section, we define a generalization of n-coherent rings for rings whose nilradical is prime.

Definition 6. Let $n \in \mathbb{N}$. A ring R is said to be a ϕ-n-coherent ring if every ϕ-n-presented module is ϕ-$(n + 1)$-presented.

Recall from [4] that a ϕ-ring R is said to be ϕ-Noetherian if $R/\mathrm{Nil}(R)$ is a Noetherian domain, which is equivalent to saying that every nonnil ideal of R is finitely generated. Recall also from [9] that the 0-coherent rings are exactly the Noetherian rings. The following result gives the analog of this result.

Proposition 3.1. *Let $R \in \mathcal{H}$. Then R is a ϕ-0-coherent ring if and only if R is a ϕ-Noetherian ring.*

Proof. Assume that R is a ϕ-0-coherent ring and let I be a nonnil ideal of R. Then R/I is a finitely generated ϕ-torsion R-module, and so R/I is a finitely presented R-module. Thus I is a finitely generated ideal. Hence R is a ϕ-Noetherian ring.

Conversely, assume that R is a ϕ-Noetherian ring and let M be a finitely generated ϕ-torsion R-module. Then M is finitely presented by [13, Theorem 3.15]. □

Recall from [9] that the 1-coherent rings are exactly the coherent rings. Proposition 3.2 gives the analog of this result.

Proposition 3.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-1-coherent ring if and only if R is a nonnil-coherent ring.*

Proof. Assume that R is a ϕ-1-coherent ring and let I be a finitely generated nonnil ideal of R. We claim that I is finitely presented. First, R/I is a finitely presented ϕ-torsion R-module, and so R/I is a ϕ-2-presented R-module. Thus I is a finitely presented ideal of R by [14, Theorem 2.1.2]. Hence R is a nonnil-coherent ring.

Conversely, assume that R is a nonnil-coherent ring and let M be a finitely presented ϕ-torsion R-module. Then $M \cong F/N$, where F is a finitely generated free R-module and N is a finitely generated submodule of F. Since R is nonnil-coherent, N is a finitely presented module by [13, Theorem 2.6]. So R is a ϕ-1-coherent ring. □

To give (counter-)examples, we use the trivial extension. Let R be a ring and E be an R-module. Then $R \propto E$, called the trivial ring extension of R by E, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in R$ and all $e, f \in E$. (This construction is also known by other terminology and other notations, such as the idealization $R(+)E$) (see [6, 14, 15, 18]).

Recall that in the classical case, if R is n-coherent, then every n-presented module is infinitely-presented. This property does not hold for the ϕ-n-coherent rings. In fact, the ring $R = \mathbb{Z} \propto \bigoplus_{i=1}^{\infty} \mathbb{Q}/\mathbb{Z}$ is an example of a ϕ-Noetherian ring, which is not nonnil-coherent by [13, Example 4.11]. So by Proposition 3.1, R is ϕ-0-coherent. However, there exists a ϕ-1-presented R-module that is not ϕ-2-presented. It follows that there exists a ϕ-0-presented R-module that is not ϕ-2-presented. Therefore, to correct this problem, in the rest of this paper we consider $(n, d) \in \mathbb{N}^2$ such that $d \leq n$.

Theorem 3.3. *Let R be a ϕ-n-coherent ring. Then every direct sum of ϕ-(n, d)-injective modules is ϕ-(n, d)-injective.*

Proof. Let R be a ϕ-n-coherent ring and let $\{N_i\}_{i \in \Gamma}$ be a family of ϕ-(n, d)-injective modules. Let I be a nonnil ideal of R such that R/I is a ϕ-n-presented module. Then R/I has a ϕ-d-presentation $F_d \to F_{d-1} \to \cdots \to F_0 \to R/I \to 0$, since $d \leq n$. Because R is ϕ-n-coherent, K_{d-1} is a finitely presented R-module, and so $\mathrm{Ext}_R^1(K_{d-1}, \bigoplus_{i \in \Gamma} N_i) \cong \bigoplus_{i \in \Gamma} \mathrm{Ext}_R^1(K_{d-1}, N_i)$ by [27, Theorem 3.9.2

(1)]. Then

$$\begin{aligned}\operatorname{Ext}_R^{d+1}(R/I,\bigoplus_{i\in\Gamma}N_i)&\cong\operatorname{Ext}_R^1(K_{d-1},\bigoplus_{i\in\Gamma}N_i)\\&\cong\bigoplus_{i\in\Gamma}\operatorname{Ext}_R^1(K_{d-1},N_i)\\&\cong\bigoplus_{i\in\Gamma}\operatorname{Ext}_R^{d+1}(R/I,N_i)\\&=0.\end{aligned}$$

Therefore, $\bigoplus_{i\in\Gamma}N_i$ is ϕ-(n,d)-injective. □

Corollary 3.4. *If R is a nonnil-coherent ring, then every direct sum of ϕ-FP-injective modules is ϕ-FP-injective.*

Proof. This follows from Propositions 2.4, 3.2 and Theorem 3.3. □

Theorem 3.5. *Every ϕ-(n,d)-ring is ϕ-n-coherent.*

Proof. If $n=0$, then the theorem is obvious from Theorem 2.22(1) and Proposition 3.1, since every ϕ-von Neumann regular ring is ϕ-Noetherian. Now, assume that $n\geq 1$ and $R\in\overline{\mathcal{H}}$. Let M be a ϕ-n-presented R-module. If M is ϕ-u-projective, then it is projective by [12, Corollary 5.36], and so M is ϕ-$(n+1)$-presented. Assume that M is not ϕ-u-projective. Then by [12, Theorem 3.10], the d-th syzygy (denoted by K) of a finite ϕ-n-presentation of M is both a finitely presented and ϕ-u-projective R-module. Again using [12, Corollary 5.36], we get that K is projective, and so M is ϕ-$(n+1)$-presented. Therefore, R is ϕ-n-coherent. □

Theorem 3.6. *Let R be a ϕ-n-coherent ring and N be an R-module. Then N is ϕ-(n,d)-injective if and only if N^+ is ϕ-(n,d)-flat.*

To prove Theorem 3.6, we need the following lemma.

Lemma 3.7. *If R is a ϕ-n-coherent ring, then for any ring T and any integer $d\geq n+1$,*

$$\operatorname{Tor}_{d+1}^R(M,\operatorname{Hom}_T(B,E))\cong\operatorname{Hom}_T(\operatorname{Ext}_R^{d+1}(M,B),E),$$

where M is a ϕ-n-presented module, E is a T-injective module, and B is an R-T-bimodule.

Proof. Assume that R is a ϕ-n-coherent ring and let M be a ϕ-n-presented module. Then M is a ϕ-d-presented module with a ϕ-d-presentation

$$F_d\to F_{d-1}\to\cdots\to F_0\to M\to 0.$$

The above exact sequence induces the exact sequence $0\to K_d\to F_d\to K_{d-1}\to 0$, and so we get the following exact sequence $\operatorname{Hom}_R(F_d,B)\to\operatorname{Hom}_R(K_d,B)\to$

$\mathrm{Ext}^1_R(K_{d-1}, B) \to 0$. Thus we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}_T(\mathrm{Ext}^1_R(K_{d-1},B),E) & \to & \mathrm{Hom}_T(\mathrm{Hom}_R(K_d,B),E) & \to & \mathrm{Hom}_T(\mathrm{Hom}_R(F_d,B),E) \\
& \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 \to & \mathrm{Tor}^R_1(K_{d-1},\mathrm{Hom}_T(B,E)) & \longrightarrow & K_d \otimes_R \mathrm{Hom}_T(B,E) & \longrightarrow & F_d \otimes_R \mathrm{Hom}_T(B,E)
\end{array}$$

Since E is a T-injective module, the two vertical right arrows are isomorphisms. Therefore, $\mathrm{Hom}_T(\mathrm{Ext}^1_R(K_{d-1}, B), E) \cong \mathrm{Tor}^R_1(K_{d-1}, \mathrm{Hom}_T(B, E))$. Moreover,

$$\begin{aligned}
\mathrm{Tor}^R_{d+1}(M, \mathrm{Hom}_T(B, E)) &\cong \mathrm{Tor}^R_1(K_{d-1}, \mathrm{Hom}_T(B, E)) \\
&\cong \mathrm{Hom}_T(\mathrm{Ext}^1_R(K_{d-1}, B), E) \\
&\cong \mathrm{Hom}_T(\mathrm{Ext}^{d+1}_R(M, B), E).
\end{aligned}$$
□

Proof of Theorem 3.6. This follows directly from Lemma 3.7 using the following isomorphism: $\mathrm{Tor}^R_{d+1}(R/I, N^+) \cong \mathrm{Ext}^{d+1}_R(R/I, N)^+$ for every nonnil ideal I of R such that R/I is a ϕ-n-presented module. □

From Proposition 3.2 and Lemma 3.7, we can obviously deduce the following Corollary 3.8.

Corollary 3.8. *Let R be a nonnil-coherent ring and M be a finitely presented ϕ-torsion module. If E is an injective R-module and B is an R-module, then we get the following isomorphism:*

$$\mathrm{Tor}^R_1(M, \mathrm{Hom}_R(B, E)) \cong \mathrm{Hom}_R(\mathrm{Ext}^1_R(M, B), E).$$

Proof. This follows immediately from Proposition 3.2 and Lemma 3.7. □

The following definition gives a generalization of ϕ-flat (resp., ϕ-FP-injective) modules.

Definition 7. Let R be a ring and $n \in \mathbb{N}^*$. An R-module M is said to be ϕ-n-flat (resp., ϕ-n-FP-injective) if M is ϕ-$(n, n-1)$-flat (resp., nonnil-$(n, n-1)$-injective).

Remark 3.9. Let M be an R-module. Then:

(1) M is ϕ-1-FP-injective if and only if M is a ϕ-FP-injective module.
(2) M is ϕ-1-flat if and only if M is a ϕ-flat module.

Next, the following result is the analog of the well-known behavior of [8, Theorem 3.1], which characterizes the ϕ-n-coherent rings.

Theorem 3.10. *Let R be a ring and $n \in \mathbb{N}^*$. Then the following are equivalent.*

(1) *R is ϕ-n-coherent.*
(2) *Every direct product of R is a ϕ-n-flat R-module.*
(3) *Every direct product of ϕ-n-flat R-modules is ϕ-n-flat.*
(4) *Every direct limit of ϕ-n-FP-injective R-modules is ϕ-n-FP-injective.*

(5) $\varinjlim \operatorname{Ext}_R^n(M, M_i) \to \operatorname{Ext}_R^n(M, \varinjlim M_i)$ *is an isomorphism for every* ϕ*-*n*-presented* R*-module* M *and every direct system* $\{M_i\}_{i\in\Gamma}$ *of* R*-modules.*

(6) $\operatorname{Tor}_n^R(\prod N_\alpha, M) \cong \prod \operatorname{Tor}_n^R(N_\alpha, M)$ *for any family* $\{N_\alpha\}$ *of* R*-modules and any* ϕ*-*n*-presented* R*-module* M*.*

(7) *An* R*-module* N *is* ϕ*-*n*-FP-injective if and only if* N^+ *is* ϕ*-*n*-flat.*

(8) *An* R*-module* N *is* ϕ*-*n*-FP-injective if and only if* N^{++} *is* ϕ*-*n*-FP-injective.*

(9) *An* R*-module* M *is* ϕ*-*n*-flat if and only if* M^{++} *is* ϕ*-*n*-flat.*

(10) $\operatorname{Tor}_n^R(M, \operatorname{Hom}_T(B, E)) \cong \operatorname{Hom}_T(\operatorname{Ext}_R^n(M, B), E)$ *for any ring* T*, where* M *is a* ϕ*-*n*-presented module,* E *is a* T*-injective module, and* B *is an* R*-*T*-bimodule.*

To prove Theorem 3.10, we need the following lemmas.

Lemma 3.11 ([8, Lemma 2.9])**.** *Let* n *be a positive integer,* A *be an* n*-presented* R*-module, and* $\{M_i\}_{i\in\Gamma}$ *be a direct system of* R*-modules (with* I *directed).*

(1) *There is an exact sequence* $0 \to \varinjlim \operatorname{Ext}_R^n(A, M_i) \to \operatorname{Ext}_R^n\left(A, \varinjlim M_i\right)$.

(2) *There is an isomorphism* $\varinjlim \operatorname{Ext}_R^{n-1}(A, M_i) \cong \operatorname{Ext}_R^{n-1}\left(A, \varinjlim M_i\right)$.

Lemma 3.12 ([8, Lemma 2.10])**.** *Let* n *be a positive integer,* A *be an* n*-presented* R*-module, and* $\{N_\alpha\}_{\alpha\in\Gamma}$ *be a family of* R*-modules.*

(1) *There is an exact sequence* $\operatorname{Tor}_n^R(\prod N_\alpha, A) \to \operatorname{Tor}_n^R(N_\alpha, A) \to 0$.

(2) *There is an isomorphism* $\operatorname{Tor}_{n-1}^R(\prod N_\alpha, A) \cong \prod \operatorname{Tor}_{n-1}^R(N_\alpha, A)$.

Proof of Theorem 3.10. (1) $\Rightarrow$ (10) This follows from Lemma 3.7.

(10) $\Rightarrow$ (7) For $B := N$, $T := \mathbb{Z}$, and $E := \mathbb{Q}/\mathbb{Z}$, we get that for every ϕ-n-presented R-module $M = R/I$, where I is a nonnil ideal of R, we have the following isomorphism $\operatorname{Tor}_n^R(M, N^+) \cong \operatorname{Ext}_R^n(M, N)^+$. So N is ϕ-n-FP-injective if and only if N^+ is ϕ-n-flat.

(7) $\Rightarrow$ (8) Let N be an R-module. If N is ϕ-n-FP-injective, then N^+ is ϕ-n-flat by hypothesis, and so N^+ is ϕ-$(n, n-1)$-flat by Definition 7. Thus N^{++} is nonnil-$(n, n-1)$-injective by Theorem 2.11. Hence N^{++} is ϕ-n-FP-injective.

Conversely, assume that N^{++} is ϕ-n-FP-injective. It follows from [26, Chapter I, Exercise 41] that N is a pure submodule of N^{++}, and so N is ϕ-n-FP-injective by Theorem 2.13.

(8) $\Rightarrow$ (9) Let M be an R-module. By Theorem 2.11 and hypothesis, M is a ϕ-n-flat module if and only if M^+ is ϕ-n-FP-injective, if and only if M^{+++} is ϕ-n-FP-injective, if and only if M^{++} is a ϕ-n-flat module.

(9) $\Rightarrow$ (3) Let $\{N_i\}_{i\in\Gamma}$ be a family of ϕ-n-flat modules. By Theorem 2.8, $\bigoplus_{i\in\Gamma} N_i$ is ϕ-n-flat, so $\left(\bigoplus_{i\in\Gamma} N_i\right)^{++} \cong \left(\prod_{i\in\Gamma} N_i^+\right)^+$ is ϕ-n-flat by hypothesis. But $\bigoplus_{i\in\Gamma} N_i^+$ is a pure submodule of $\prod_{i\in\Gamma} N_i^+$ by [7, Lemma 1 (1)], and so $\left(\prod_{i\in\Gamma} N_i^+\right)^+ \to \left(\bigoplus_{i\in\Gamma} N_i^+\right)^+ \to 0$ splits. Thus $\prod_{i\in\Gamma} N_i^{++} \cong \left(\bigoplus_{i\in\Gamma} N_i^+\right)^+$, and so $\prod_{i\in\Gamma} N_i^{++}$ is ϕ-n-flat. Since $\prod_{i\in\Gamma} N_i$ is a pure submodule of $\prod_{i\in\Gamma} N_i^{++}$ (see [7, Lemma 1 (2)]), $\prod_{i\in\Gamma} N_i$ is ϕ-n-flat by Theorem 2.13.

(3) $\Rightarrow$ (2) This is straightforward.

(2) $\Rightarrow$ (1) Let M be a ϕ-n-presented with a ϕ-n-finite presentation $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$. We claim that $K_{n-1} := \ker(F_{n-1} \longrightarrow F_{n-2})$ is a finitely presented R-module. First, we have the following exact sequence $0 \to K_{n-1} \to F_{n-1} \to K_{n-2} \to 0$. Let I be an indexing set. Then K_{n-2} is finitely presented, since M is ϕ-n-presented, and so $R^I \otimes_R K_{n-2} \cong K^I_{n-2}$ from [26, Lemma 13.2]. From the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_{n-1} \otimes_R R^I & \longrightarrow & F_{n-1} \otimes_R R^I & \longrightarrow & K_{n-2} \otimes_R R^I \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & K^I_{n-1} & \longrightarrow & F^I_{n-1} & \longrightarrow & K^I_{n-2},
\end{array}$$

it follows that K_{n-1} is finitely presented, and so M is ϕ-$(n+1)$-presented. Thus R is ϕ-n-coherent.

(1) $\Rightarrow$ (5) This follows immediately from Lemma 3.11(2).

(5) $\Rightarrow$ (4) This is straightforward.

(4) $\Rightarrow$ (1) Let M be a ϕ-n-presented module with a ϕ-n-finite presentation

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0.$$

We claim that $K_{n-1} := \ker(F_{n-1} \longrightarrow F_{n-2})$ is a finitely presented R-module. Let $\{N_i\}_{i \in \Gamma}$ be a family of injective modules. Then $\varinjlim N_i$ is ϕ-n-FP-injective by hypothesis. Hence, $\mathrm{Ext}^1_R(K_{n-2}, \varinjlim N_i) \cong \mathrm{Ext}^n_R(M, \varinjlim N_i) = 0$, and so we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(K_{n-2}, \varinjlim N_i) & \longrightarrow & \mathrm{Hom}_R(F_{n-1}, \varinjlim N_i) & \longrightarrow & \mathrm{Hom}_R(K_{n-1}, \varinjlim N_i) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\varinjlim \mathrm{Hom}_R(K_{n-2}, N_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F_{n-1}, N_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(K_{n-1}, N_i) & \longrightarrow & 0.
\end{array}$$

Therefore, the left two vertical arrows are isomorphisms by [20, Satz 3], and so $\mathrm{Hom}_R(K_{n-1}, \varinjlim N_i) \cong \varinjlim \mathrm{Hom}_R(K_{n-1}, N_i)$. Thus K_{n-1} is finitely presented by [17, Proposition 2.5], and so M is ϕ-$(n+1)$-presented. Therefore, R is a ϕ-n-coherent ring.

(1) $\Rightarrow$ (6) This follows from Lemma 3.12(2).

(6) $\Rightarrow$ (3) This is straightforward. □

By Proposition 3.2 and Theorem 3.10, we can immediately deduce the following result, which characterizes nonnil-coherent rings.

Corollary 3.13. *The following statements are equivalent for a ϕ-ring R.*

(1) *R is a nonnil-coherent ring.*
(2) *Any direct product of R is a ϕ-flat R-module.*
(3) *Any direct product of ϕ-flat R-modules is ϕ-flat.*
(4) *Every direct limit of ϕ-FP-injective R-modules is ϕ-FP-injective.*

(5) $\varinjlim \operatorname{Ext}^1_R(M, M_i) \to \operatorname{Ext}^1_R(M, \varinjlim M_i)$ *is an isomorphism for every finitely presented* ϕ*-torsion* R*-module* M *and every direct system* $\{M_i\}_{i\in\Gamma}$ *of* R*-modules.*
(6) $\operatorname{Tor}^R_1(\prod N_\alpha, M) \cong \prod \operatorname{Tor}^R_1(N_\alpha, M)$ *for any family* $\{N_\alpha\}$ *of* R*-modules and any finitely presented* ϕ*-torsion* R*-module* M*.*
(7) *An* R*-module* N *is* ϕ*-FP-injective if and only if* N^+ *is* ϕ*-flat.*
(8) *An* R*-module* N *is* ϕ*-FP-injective if and only if* N^{++} *is* ϕ*-FP-injective.*
(9) *An* R*-module* M *is* ϕ*-flat if and only if* M^{++} *is* ϕ*-flat.*
(10) $\operatorname{Tor}^R_1(M, \operatorname{Hom}_T(B, E)) \cong \operatorname{Hom}_T(\operatorname{Ext}^1_R(M, B), E)$ *for any ring* T*, where* M *is a finitely presented* ϕ*-torsion module,* E *is a* T*-injective module, and* B *is an* R*-*T*-bimodule.*

Acknowledgements. The authors would like to thank the reviewer for his/her comments. H. Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education (2021R1I1A3047469).

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